JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 26, No. 4, July–August 2003

Engineering Notes

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Instability of Fixed, Low-Thrust Drag Compensation

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Introduction

FORCED drag compensation using continuous low-thrust propulsion has been considered for satellites in low Earth orbit. This simple, but nonoptimal, scheme merely requires that the thrust vector is directed opposite to the drag vector and that the magnitude of the two are equal. In principle, the drag force acting on the spacecraft could be determined onboard using accurate accelerometers. However, for small, low-cost spacecraft such sensors may be unavailable. An alternative strategy would be to fix the thrust magnitude equal to the expected air drag that would be experienced by the spacecraft. The thrust level would be periodically updated based on ground-based orbit determination. In this Engineering Note, it is shown that such a forced circular orbit with a fixed thrust level is exponentially unstable for all physically reasonable atmosphere models.

Forced Drag Compensation

For a circular low Earth orbit, the dominant perturbing effects are due to atmospheric drag and Earth oblateness.[1] Atmospheric drag perturbations result in a secular decrease in orbital radius r due to frictional energy loss. To fix the orbit radius, one possible scheme is to compensate directly for air drag $\boldsymbol{D}$ by applying an opposite, continuous low thrust $\boldsymbol{T}$. The equation of motion for a spacecraft of mass m is then

$$\ddot{\boldsymbol{r}} + \mu(\boldsymbol{r}/r^3) = (1/m)(\boldsymbol{T} - \boldsymbol{D}) \tag{1}$$

which shows that if $\boldsymbol{T} = \boldsymbol{D}$ an ideal two-body orbit will be obtained, where μ is the gravitational parameter of the problem. When the scalar product of Eq. (1) is taken with the spacecraft velocity vector $\boldsymbol{v}$, it is found that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} - \frac{\mu}{r}\right] = \frac{1}{m}(\boldsymbol{T} - \boldsymbol{D})\cdot\boldsymbol{v} \tag{2}$$

where the term in brackets of the left side is just the total orbit energy E. For a circular orbit, the total orbit energy is $E = -\mu/2r$. In addition, the local circular orbit speed $\|\boldsymbol{v}\| = \sqrt{(\mu/r)}$ so that Eq. (2) may be used to obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2r^{\frac{3}{2}}}{\sqrt{\mu}}\frac{\|\boldsymbol{T} - \boldsymbol{D}\|}{m} \tag{3}$$

The general form of the air drag acting on the spacecraft is given by

$$\|\boldsymbol{D}\| = \tfrac{1}{2}C_D A\rho(r)\boldsymbol{v}\cdot\boldsymbol{v} \tag{4}$$

where A is the aerodynamic reference area, C_D is the drag coefficient, and ρ is the air density at orbit radius r. Again when it is noted that $\|\boldsymbol{v}\| = \sqrt{(\mu/r)}$ and constant thrust $\|\boldsymbol{T}\|$ is assumed, the circular orbit radius evolves according to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2\|\boldsymbol{T}\|}{\sqrt{\mu}m}r^{\frac{3}{2}} - \frac{\sqrt{\mu}C_D A}{m}\rho(r)r^{\frac{1}{2}} \tag{5}$$

For a given operating orbit radius $\tilde{r}$, the required thrust to provide equilibrium is then given by

$$\|\boldsymbol{T}\| = (C_D A\mu/2\tilde{r})\rho(\tilde{r}) \tag{6}$$

With use of this fixed thrust, the stability of the resulting forced circular orbit will be investigated by linearization. Note that the forced circular orbit is in fact nonoptimal in terms of fuel consumption.[2]

Linear Orbit Instability

To determine the stability of the equilibrium operating orbit, the spacecraft orbit radius will be written as $r = \tilde{r} + \xi$. Then, expanding Eq. (5) to first order, with the required thrust provided by Eq. (6), yields

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \Lambda(\tilde{r})\xi \tag{7}$$

where the constant coefficient $\Lambda(\tilde{r})$ is found to be

$$\Lambda(\tilde{r}) = \left(\sqrt{\mu}C_D A\big/m\right)\left[\rho(\tilde{r})\big/\tilde{r}^{\frac{1}{2}}\right]\{1 - \tilde{r}[\rho'(\tilde{r})/\rho(\tilde{r})]\} \tag{8}$$

and where $\rho'(r) = \mathrm{d}\rho/\mathrm{d}r$. The condition required for stability is that $\Lambda(\tilde{r}) \prec 0$, which in turn requires that

$$\rho'(\tilde{r})/\rho(\tilde{r}) \succ 1/\tilde{r} \tag{9}$$

If an exponential atmosphere is now assumed with scale height H and base density ρ_0, so that

$$\rho(r) = \rho_0 \exp[-(r - R)/H] \tag{10}$$

where R is the radius of the Earth, the condition for stability then becomes

$$-1/H \succ 1/\tilde{r} \tag{11}$$

Because $\tilde{r} \succ 0$, this then implies that $H \prec 0$, which is clearly unphysical. In the general case, if $\rho'(r) \sim \Delta\rho/\Delta r$, then Eq. (9) implies that

$$\Delta\rho/\rho \succ \Delta r/\tilde{r} \tag{12}$$

so that if $\Delta r \succ 0$, stability requires $\Delta\rho \succ 0$, which again is clearly unphysical in any reasonable atmosphere. The phase space of Eq. (5)

Received 15 July 2002; revision received 19 February 2003; accepted for publication 19 February 2003.

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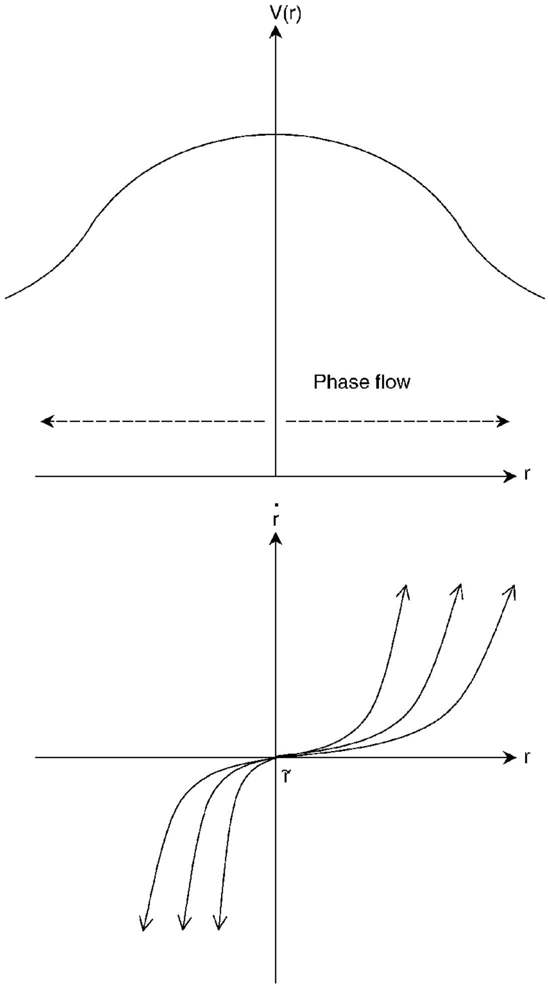

Fig. 1 Schematic potential and phase space.

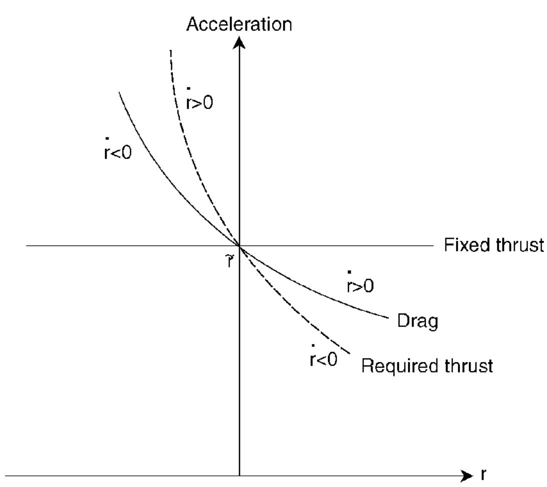

Fig. 2 Conditions for stability.

is shown schematically in Fig. 1. Note that such an unphysical atmosphere model has in fact been considered to illustrate some subtle points in optimal control theory.[3]

Physically, this instability is due to the interaction between the two accelerations acting on the satellite. Figure 2 shows a schematic diagram of the variation of drag and low-thrust acceleration with altitude. For altitudes higher than the equilibrium altitude, the low-thrust acceleration exceeds the drag acceleration so that the satellite spirals outward. Conversely, for altitudes lower than the equilibrium altitude, the drag acceleration exceeds the low-thrust acceleration, and the satellite spirals inward. It can be seen that to stabilize the equilibrium operating altitude, the low-thrust acceleration would be required to be shaped as a function of altitude, as shown in Fig. 2.

Nonlinear Orbit Instability

Although it has been demonstrated that the forced circular orbit is linearly unstable, which is both a necessary and sufficient condition for nonlinear instability, it is interesting to investigate the nonlinear stability properties of Eq. (5). For ease of analysis, Eq. (5) will be considered with the exponential atmosphere model defined by Eq. (10) to obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha r^{\frac{3}{2}} - \beta\sqrt{r}\exp\left(\frac{-r}{H}\right), \qquad \alpha = \frac{2\|\boldsymbol{T}\|}{\sqrt{\mu}m}$$

$$\beta = \frac{\sqrt{\mu}C_D A}{m}\exp\left(\frac{R}{H}\right) \tag{13}$$

This first-order dynamic system may now be written as a function of some potential $V(r)$ such that[4]

$$\dot{r} = -\frac{\partial V(r)}{\partial r} \tag{14}$$

where the potential is found to be

$$V(r) = -\tfrac{2}{5}\alpha r^{\frac{5}{2}} - \beta H^{\frac{3}{2}}\Gamma\left[\tfrac{3}{2}, r/H\right] \tag{15}$$

with Γ the incomplete gamma function. It can be seen that $V(r)$ has a turning point, $V'(r) = 0$ at $r = \tilde{r}$, when

$$\alpha/\beta = (1/\tilde{r})\exp(-\tilde{r}/H) \tag{16}$$

which is equivalent to Eq. (6). Note that, for a given pair of parameters α and β, the equilibrium altitude can be determined as a solution to Eq. (16). Although implicit in $\tilde{r}$, Eq. (16) can be solved using the Lambert W function (see Ref. 5), the solution of which is implemented in symbolic mathematics packages as the ProductLog function. This function is defined such that $W(z)$ returns the principal solution of $z = We^W$, with $W(z)$ real if $z \succ -1/e$, and can be viewed as an extension of the usual logarithm function. The function also satisfies the differential equation $\mathrm{d}W/\mathrm{d}z + W/z(1+W)$. When this function is used, it can be shown that Eq. (16) provides the equilibrium altitude $\tilde{r}$ as

$$\tilde{r}/H = W(\beta/\alpha H) \tag{17}$$

To demonstrate that $\tilde{r}$ is unstable in general, it must be shown that $\tilde{r}$ corresponds to a single global maximum of the potential $V(r)$. Therefore, calculating $V''(r)$ as

$$V''(r) = -\tfrac{3}{2}\alpha\sqrt{r} + \left(\beta\big/2\sqrt{r}\right)(1 - 2r/H)\exp(-r/H) \tag{18}$$

it can be seen that because $\tilde{r} \succ H/2$ then $V''(\tilde{r}) \prec 0$ so that the equilibrium orbit radius $\tilde{r}$ corresponds to a maximum in the potential $V(r)$ and indeed $V''(r) \prec 0$ for $r \succ H/2$. It is then sufficient to note that $V'(r) = 0$ has no solutions for $r \succ 0$, other than that defined by Eq. (16). Therefore, the equilibrium orbit radius $\tilde{r}$ corresponds to a single global maximum in the potential $V(r)$ and so the equilibrium altitude has nonlinear instability, as shown in Fig. 1.

Conclusions

It has been shown that a forced circular orbit, using a fixed low-thrust acceleration to compensate for air drag, is exponentially unstable. The instability has been determined using both a linear and nonlinear analysis. For the linear analysis, it was demonstrated that the instability exists for all reasonable atmosphere models, whereas the nonlinear analysis assumed an exponential atmosphere model. Because the linear instability condition is both necessary and sufficient, however, it is determined that the instability is independent of the atmosphere model assumed. A stable forced orbit would require a montonically increasing air density with orbit radius, which is clearly unphysical.

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Linearized Dynamic Equations for Spacecraft Subject to J_2 Perturbations

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Introduction

THE dynamic equations of motion for a spacecraft relative to a neighboring orbit have taken on added importance in the last few years due to the possibility of spacecraft orbiting in formation. For a number of mathematical and physical reasons, linearized dynamic equations are of interest. The survey by Carter[1] shows that there are a number of ways to linearize the equations of motion for a spacecraft. Of particular interest is the so-called Hill's equation[2] a somewhat unfortunate name because "Hill's equation is a convenient abbreviation defining the class of homogeneous, linear, second-order differential equations with real, periodic coefficients."[3] For this and other reasons, the linearized orbital equations are more commonly known as the Clohessy–Wiltshire (C–W) equations.[4] In any case, these equations are typically used for an inverse-square gravity field, although the inclusion of a perturbation acceleration allows them to be used, in principle, for more general cases. As shown by Alfriend et al.[5] and others, the effect of a J_2 perturbation gradually destroys the configuration of the formation of the orbiting spacecraft. Consequently, a number of propellant-reduction formation-keeping strategies are proposed as summarized by Vadalli et al.[6] To better understand the effects of J_2 and eccentricity, Gim and Alfriend[7] obtain state transition matrices by way of various similarity transformations. From this brief review of the literature, it is apparent that a set of linearized dynamic equations that can handle the effects of J_2 would be useful in further analyzing the dynamics of spacecraft in formation.

In this Note, we present a new set of linearized dynamic equations that account for the J_2 perturbations. We limit the results for the case of a circular reference orbit. In principle, these equations can be generalized for elliptical reference orbits as described by Melton.[8]

Main Result

We present the main result here and derive it in the following section. For clarity, we define a reference frame O as one that is centered at a radial distance R_0 from the center of the Earth, whose origin moves at a circular speed $[=\sqrt{(\mu/R_0)}]$, while the frame rotates at a circular angular velocity $[=n=\sqrt{(\mu/R_0^3)}]$ about its number 3 axis such that the number 1 axis is along the outward R_0 direction. In other words, this frame is equivalent to a Frenet frame associated with a hypothetical spacecraft orbiting Earth in an inverse-square gravity field, that is, with its J_2 component removed. Let, $\boldsymbol{r}$ be the position vector of a (real) spacecraft in this coordinate system. When $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are the first and second time derivatives of $\boldsymbol{r}$ in O, the linearized equations of motion with J_2 effects can be written as

$$\ddot{\boldsymbol{r}} + C\dot{\boldsymbol{r}} + [K + P(t)]\boldsymbol{r} + \boldsymbol{q}(t) = \boldsymbol{a} \tag{1}$$

where $\boldsymbol{a}$ is the perturbing acceleration on the spacecraft (resulting, for example, from the higher-order gravitational harmonics, thrust, drag, etc.) and the other quantities are defined as follows: C and K are 3×3 constant matrices,

$$C = 2n\begin{bmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \qquad K = n^2\begin{bmatrix} -3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{2}$$

$P(t)$ is a 3×3 symmetric matrix periodic function that can be written as

$$P(t) = n^2 J_R\begin{bmatrix} 12\sin^2 i\,\sin^2(nt) - 4 & -4\sin^2 i\,\sin(2nt) & -4\sin(2i)\sin(nt)\\ -4\sin^2 i\,\sin(2nt) & 1+\sin^2 i(2-7\sin^2(nt)) & \sin(2i)\cos(nt)\\ -4\sin(2i)\sin(nt) & \sin(2i)\cos(nt) & 3-\sin^2 i(2+5\sin^2(nt))\end{bmatrix} \tag{3}$$

where i is the inclination of the hypothetical spacecraft, t is the time computed from passage of the ascending node, that is, the clock starts when the hypothetical spacecraft crosses the ascending node, and J_R is a nondimensional number given by

$$J_R = \frac{3J_2R_\oplus^2}{2R_0^2} \tag{4}$$

Last, $\boldsymbol{q}(t)$ is the 3×1 vector periodic function

$$\boldsymbol{q}(t) = n^2 J_R R_0\begin{bmatrix} 1-3\sin^2(nt)\sin^2 i\\ \sin(2nt)\sin^2 i\\ \sin(nt)\sin(2i)\end{bmatrix} \tag{5}$$

Remark 1: It is obvious that for $J_2 = 0$, Eq. (1) is the familiar C–W equation. For $J_2 \neq 0$, Eq. (1) belongs to the class of Hill's equation (see Ref. 3).

Remark 2: The equations are deceptive in the sense that it appears that the J_2-perturbed motion of the spacecraft does not depend on Ω, the right ascension of the ascending node. This interpretation is not true because the equations of motion of the spacecraft are written in a frame that is deliberately chosen to be unaffected by J_2. That is, the orbital elements in Eq. (1) are that of the static reference orbit and not equal to the dynamic (osculating) elements of the spacecraft except possibly at the time of initialization.

Remark 3: The simplest case is zero inclination. In this situation, $\boldsymbol{q}(t)$ is a constant with its last two elements being zero, whereas the P matrix simplifies to a constant diagonal matrix. Note that these equations are not singular for zero inclination.

Received 20 November 2001; revision received 24 February 2003; accepted for publication 10 March 2003.

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